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D-dimensional developed MHD turbulence: double expansion model

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Abstract
Developed magnetohydrodynamic turbulence near two dimensions d up to three dimensions has been investigated by means of a renormalization group approach and double expansion regularization. A modification of a standard minimal subtraction scheme has been used to analyse the stability of the Kolmogorov scaling regime which is governed by the renormalization group fixed point. The exact analytical expressions have been obtained for the fixed points. The continuation of the universal value of the inverse Prandtl number $u = 1.562$ determined at $d = 2$ up to $d = 3$ restores the value of $u = 1.393$ which is known in the kinetic fixed point from the usual ϵ-expansion. The magnetic stable fixed point has been calculated and its stability region has also been examined. This point loses stability: (a) below a critical value of dimension $d_c = 2.36$ (independently of the a-parameter of a magnetic forcing) and (b) below the value of $a_c = 0.146$ (independently of the dimension).

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1. Introduction

The renormalization group (RG) methods have been widely used for the analysis of fully developed hydrodynamic (HD) turbulence beginning from pioneering papers [1, 2] based on [3, 4]. It gives a possibility of replying upon some principal questions, e.g., on the fundamental description of the infrared (IR) scale invariance; as well as it is useful for calculation of many quantities, e.g., critical dimensions of fields and their gradients, viscosity, etc (see, e.g., [5–8]).

Then many authors begin to use Wilson's scheme or some adequate generalized renormalization scheme to study the HD turbulence [9] as well as magnetohydrodynamic (MHD) turbulence [10, 11]. This time Vasiliev's team have used functional formulation of the

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field-theoretic RG [12, 13] to legalize the Kolmogorov scaling regime of the HD turbulence [14, 15]. They consider (as used in the present paper) the functional quantum field RG approach [5] rather than Wilson's RG technique [16]. It assigns a field action to the stochastic problem and makes possible to use an elegant and very well-developed RG procedure in quantum field theory to investigate infrared asymptotic regimes of a stochastic system. Then this RG method has been applied in the MHD turbulence [17, 18]. Note here that this functional RG method allows a straightforward extension of the perturbative calculation to higher order loops without a principal difficulty (see [19, 20], for example).

Considerable effort had been devoted to the application of adequate field-theoretic methods in the MHD turbulence (see a recent review of Verma [21], for example). Authors in [10, 11] have used the 'classical' Yakhot–Orszag scheme [9]. In the past few years Verma [22, 23] performed detailed RG calculation of the MDH turbulence using McComb's alternative field-theoretic RG procedure [7] and reached a notable progress in the calculation of some renormalized parameters of the MHD turbulence. Here we will not present full discussion of all methods used in the fully developed turbulence theory such as calculation of the Alfven ratio, magnetic resistivity [23] or a problem of magnetic dynamo in helicit MHD [18] because it goes out of the frame of the present paper (but see some remarks and discussion in sections 4 and 5).

The present paper deals with an investigation of the existence and range of stability of the 'magnetic' scaling regime (i.e. the magnetic fixed point for the zero inverse Prandtl number, see below) in the non-helical d-dimensional MHD turbulence. The existence of two different anomalous scaling regimes in three dimensions, which are known as kinetic and magnetic regimes, was established in the pioneering papers [10, 17]. These two points correspond to two IR stable fixed points of the RG. On the other hand, it was also supposed that in two dimensions the magnetic fixed point does not exist as a result of nonexistence of the IR stable magnetic fixed point. But the conclusions about two-dimensional fixed points cannot be considered without doubts in these papers due to the problems with renormalization in two dimensions which were not taken into account [24] (see also [5]). In [25, 26] the two-dimensional case was studied too, but again with shortcomings; therefore, their results cannot be considered completely conclusive. Within our field-theoretic RG approach the problem is related to the existence of additional divergences which arise in two dimensions.

The first correct treatment of the two-dimensional case of the stochastically forced MHD equations with the proper account of these additional divergences was done by the authors of [27]. It was accomplished within a two-parameter expansion (double expansion) of scaling exponents and scaling functions [24] where, besides the parameter which characterizes the deviation of the exponent of the powerlike correlation function of random forcing from its critical value, the additional parameter of the deviation of the spatial dimension was introduced. The use of this double expansion method has allowed them to confirm the basic conclusions of the previous works [10, 17], namely, the nonexistence of the magnetic scaling regime near two dimensions.

The authors of the paper [27] also tried to restore the stability of the magnetic fixed point when moving from two dimensions in the direction of three dimensions. This possibility was achieved by using the special choice of finite renormalization which allowed them to keep track of the effect of the additional divergences near two dimensions. Technically, it was done by introducing another uniform UV cutoff in all propagators which does not affect the large-scale properties of the model. This setup is similar to that of Polchinski [28]. As a result, the borderline dimension between the stable and unstable magnetic fixed points was found and it leads to the possibility of the uniform description of two- and three-dimensional cases of stochastic MHD.

Another possibility of solving the problem of the additional divergences in two dimensions together with the problem of restoration of the stability of the corresponding fixed point when going from a two-dimensional system to a three-dimensional one was proposed by the authors of [29]. They suggest to apply a modified minimal subtraction (MS) scheme in which the d-dependence of the tensor structures of the UV divergent parts of the corresponding diagrams is kept. It was successfully used in the fully developed Navier–Stokes turbulence with weak uniaxial anisotropy to restore the stability of the Kolmogorov scaling regime which is unstable in two dimensions and stable in three dimensions.

In what follows, we shall apply the double expansion method together with the modified MS scheme introduced in [29] to the stochastic MHD equations. Our aim is to investigate whether it is possible to describe correctly and uniformly the two- dimensional and the three-dimensional systems and to compare our results with those of [27] where the different method was used (see above). Thus, we carry out an analysis of the randomly forced MHD equations with the proper account of the additional UV divergences which are appeared in $d = 2$. We apply a modified minimal subtraction scheme based on the fact that the tensor structure of counter-terms is left generally d-dependent in the calculations of divergent parts of Green's functions. It will be shown that it allows us to investigate the behaviour of the system under continual transition to $d = 3$ beginning from $d = 2$. We have also confirmed the earlier conclusions made in [10, 17, 22] that near two dimensions a scaling regime driven by the velocity fluctuations may exist, but no magnetically driven scaling regime can occur. We have also investigated the long-range asymptotic behaviour of the model in the double expansion framework and found, in particular, that in this case thermal fluctuations of the magnetic scaling regime may occur and that the value of the borderline dimension is significantly lower ($d_c = 2.36$) than that in the ϵ-expansion [10] ($d_c = 2.85$) and also lower than that in the 'modified' double expansion introduced in [27] ($d_c = 2.46$), but it is rather higher than the value ($d_c = 2.2$) calculated in the frame of McComb's renormalization [21]. The discrepancy between the value of the inverse Prandtl number u which corresponds to the nontrivial stable fixed point of the RG in three dimensions, which has been obtained in the double expansion scheme in earlier paper [30] and that obtained by the usual ϵ-expansion scheme [10, 17] and also that obtained by Verma [22, 23] by McComb's procedure, was one more reason of the present analysis. Here we show that the continuous transition from the universal value of the inverse Prandtl number $u = 1.562$ determined at $d = 2$ restores the value of $u = 1.393$ at $d = 3$ which is known from the usual ϵ-expansion.

The paper is organized as follows: in section 2 the functional field-theoretic formulation of the model is presented in detail. In section 3 the renormalization of the model is discussed. In section 4 a detailed analysis of the possible scaling regimes is done. In section 5 conclusions and discussion of the results are given.

2. Functional formulation of the double expansion model

In the present paper we study the universal statistical features of the model of stochastic MHD which is described by the system of equations for the fluctuating velocity field of an incompressible conducting fluid $\boldsymbol{v}(x), x \equiv (\boldsymbol{x}, t), \nabla \cdot \boldsymbol{v} = 0$ and the magnetic induction $\mathbf{B} = (\rho\mu)^{1/2}\boldsymbol{b}(x)$ (where ρ is the density of the fluid and μ is its permeability) [10, 17, 31]:

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{b} - \nu_0\nabla^2\boldsymbol{v} = \boldsymbol{f}^v, \qquad (1)$$

$$\partial_t \boldsymbol{b} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{b} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{v} - \nu_0 u_0\nabla^2\boldsymbol{b} = \boldsymbol{f}^b, \qquad (2)$$

with the incompressibility conditions $\nabla\cdot\boldsymbol{f}^v = 0$ and $\nabla\cdot\boldsymbol{f}^b = 0$ and the field $\boldsymbol{b}$ is supposed to be solenoidal too, $\nabla\cdot\boldsymbol{b} = 0$. The statistics of $\boldsymbol{v}, \boldsymbol{b}$ is completely determined by both the nonlinear equations (1), (2) and the statistics of the external inter-correlated large-scale random forces $\boldsymbol{f}^v$, $\boldsymbol{f}^b$. The dissipation is controlled by the parameter of the kinematic viscosity ν_0, and u_0 denotes the inverse Prandtl number (hereafter all parameters with a subscript 0 denote bare parameters of unrenormalized theory; see below). Note here that the term $(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}$ expresses the transverse part of the Lorentzian force and it can be omitted in the case of a magnetic field treated as a passive admixture.

As usual [10, 17], statistical properties of the Gaussian forcing with zero mean values ($\langle\boldsymbol{f}^v\rangle = 0$, $\langle\boldsymbol{f}^b\rangle = 0$) are determined by the relations

$$\left\langle f_j^v(1) f_s^v(2)\right\rangle = \delta(\tau) u_0 \nu_0^3 \int \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d} \mathrm{P}_{js}(\boldsymbol{k})\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} [g_{v10} k^{2-2\delta-2\epsilon} + g_{v20} k^2], \tag{3}$$

$$\left\langle f_j^b(1) f_s^b(2)\right\rangle = \delta(\tau) u_0^2 \nu_0^3 \int \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d} \mathrm{P}_{js}(\boldsymbol{k})\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} [g_{b10} k^{2-2\delta-2a\epsilon} + g_{b20} k^2], \tag{4}$$

where the argument $1 \equiv x_1$, $\tau = t_1 - t_2$, $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$, $\mathrm{P}_{js}(\boldsymbol{k}) = \delta_{js} - k_j k_s/k^2$, the parameter ϵ determines the powerlike falloff of the long-range forcing correlations, and the parameter $\delta = (d-2)/2$ describes the deviation from the spatial dimension $d = 2$. The free parameter a controls the power form of the magnetic forcing. Note that the parameters $\epsilon = 2$, $a = 1$ are the natural 'physical' values in our 'massless' power-law energy injection. The introduction of the local correlations (proportional to the new couplings g_{v20} and g_{b20}) which are described by the analytic in k^2 terms in the correlation functions (3) and (4), is related to the existence of additional divergences of this structure (see below in the text) in the two-dimensional model which cannot be removed by corresponding nonlocal terms [24, 32, 33]. At the same time, the localness of the counter-terms is the fundamental feature of a model to be multiplicatively renormalizable [13, 34]. For example, it was not taken into account in the analysis of the model in [10, 17].

Using the well-known Martin–Siggia–Rose formalism [3, 4], one can transform the stochastic problem (1)–(2) with correlators (3) and (4) into the field-theoretic model of the doubled set of fields $\Phi \equiv \{\boldsymbol{v}, \boldsymbol{b}, \boldsymbol{v}', \boldsymbol{b}'\}$ with the following action functional:

$$\begin{aligned} S = \frac{1}{2}\int \mathrm{d}x_1\, \mathrm{d}x_2 \big\{ v_j'(1)\big\langle f_j^v(1) f_s^v(2)\big\rangle_0 v_s'(2) + b_j'(1)\big\langle f_j^b(1) f_s^b(2)\big\rangle_0 b_s'(2)\big\} \\ + \int \mathrm{d}x\, \boldsymbol{v}' \cdot (-\partial_t \boldsymbol{v} + \nu_0 \nabla^2 \boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{b}) \\ + \int \mathrm{d}x\, \boldsymbol{b}' \cdot (-\partial_t \boldsymbol{b} + u_0\nu_0 \nabla^2 \boldsymbol{b} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{b}), \end{aligned} \tag{5}$$

where $\boldsymbol{v}'$ and $\boldsymbol{b}'$ are independent of $\boldsymbol{v}$ and $\boldsymbol{b}$ auxiliary incompressible fields, which we have to introduce when transforming the stochastic problem into a functional form.

The dimensional constants g_{v10}, g_{b10}, g_{v20} and g_{b20}, which control the amount of randomly injected energy given by (3), (4), play the role of the coupling constants in the perturbative expansion. For the convenience of further calculations the factors $\nu_0^3 u_0$ and $\nu_0^3 u_0^2$ including the 'bare' (molecular) viscosity ν_0 and the 'bare' (molecular or microscopic) magnetic inverse Prandtl number u_0 have been extracted. As was already mentioned the bare (non-renormalized) quantities are denoted by the subscript '0'.

The most important measurable quantities in the study of a fully developed turbulence and related problems are considered to be the statistical objects represented by correlation and response functions (Green functions) of the fields. Standardly, the formulation through the action functional (5) replaces the statistical averages of random quantities in the stochastic problem (1)–(4) with equivalent functional averages with weight $\exp S(\Phi)$. Generating functionals of total Green functions $G(A)$ and connected Green functions $W(A)$ are then defined by the functional integral

$$G(A) = \mathrm{e}^{W(A)} = \int \mathcal{D}\Phi\, \mathrm{e}^{S(\Phi)+A\Phi}, \tag{6}$$

where $A(x) = \{\mathbf{A}^{\boldsymbol{v}}, \mathbf{A}^{\boldsymbol{b}}, \mathbf{A}^{\boldsymbol{v}'}, \mathbf{A}^{\boldsymbol{b}'}\}$ represents a set of arbitrary sources for the set of fields Φ, $\mathcal{D}\Phi \equiv \mathcal{D}\theta\mathcal{D}\theta'\mathcal{D}\mathbf{v}\mathcal{D}\mathbf{v}'$ denotes the measure of functional integration, and the linear form $A\Phi$ is defined as

$$A\Phi = \int \mathrm{d}x [\mathbf{A}^{\boldsymbol{v}}(x)\cdot\boldsymbol{v}(x) + \mathbf{A}^{\boldsymbol{b}}(x)\cdot\boldsymbol{b}(x) + \mathbf{A}^{\boldsymbol{v}'}(x)\cdot\boldsymbol{v}'(x) + \mathbf{A}^{\boldsymbol{b}'}(x)\cdot\boldsymbol{b}'(x)]. \tag{7}$$

The functional formulation gives the possibility of using the field-theoretic methods, including the RG technique, to solve the problem. By means of the RG approach it is possible to extract the large-scale asymptotic behaviour of the correlation functions after an appropriate renormalization procedure which is needed to remove UV divergences. The functional formulation is advantageous also because the Green functions of the Fourier-decomposed stochastic MHD can be calculated by means of a Feynman diagrammatic technique.

Action (5) is given in a form convenient for a realization of the field-theoretic perturbation analysis with the standard Feynman diagrammatic technique. Free (bare) propagators $\hat{\Delta}$ can be found from the quadratic part of the action (5) written in the form $-(1/2)\Phi\hat{\mathcal{K}}\Phi$ and by using the definition $\hat{\mathcal{K}}\hat{\Delta} = \hat{1}$, where $\hat{1}$ denotes the diagonal matrix whose diagonal elements are the transverse projectors (our fields are solenoidal). One obtains

$$\hat{\Delta}_{js} = \begin{pmatrix} \Delta^{vv}_{js} & 0 & \Delta^{vv'}_{js} & 0 \\ 0 & \Delta^{bb}_{js} & 0 & \Delta^{bb'}_{js} \\ \Delta^{v'v}_{js} & 0 & 0 & 0 \\ 0 & \Delta^{b'b}_{js} & 0 & 0 \end{pmatrix} \tag{8}$$

with the elements (wave-number-frequency representation)

$$\begin{aligned}
\Delta^{vv'}_{js}(\mathbf{k},\omega) &= \Delta^{v'v}_{js}(-\mathbf{k},-\omega) = \frac{\mathrm{P}_{js}(\mathbf{k})}{-\mathrm{i}\omega + \nu_0 k^2}, \\
\Delta^{bb'}_{js}(\mathbf{k},\omega) &= \Delta^{b'b}_{js}(-\mathbf{k},-\omega) = \frac{\mathrm{P}_{js}(\mathbf{k})}{-\mathrm{i}\omega + u_0\nu_0 k^2}, \\
\Delta^{vv}_{js}(\mathbf{k},\omega) &= u_0\nu_0^3 k^2 \frac{g_{v10}k^{-2\delta-2\epsilon} + g_{v20}}{|-\mathrm{i}\omega + \nu_0 k^2|^2}\mathrm{P}_{js}(\mathbf{k}), \\
\Delta^{bb}_{js}(\mathbf{k},\omega) &= u_0^2\nu_0^3 k^2 \frac{g_{b10}k^{-2a\delta-2\epsilon} + g_{b20}}{|-\mathrm{i}\omega + u_0\nu_0 k^2|^2}\mathrm{P}_{js}(\mathbf{k}).
\end{aligned} \tag{9}$$

The model has three triple (interaction) vertices

$$-\boldsymbol{v}'(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = v'_j V_{jkl} v_k v_l, \tag{10}$$

$$-\boldsymbol{v}'(\boldsymbol{b}\cdot\nabla)\boldsymbol{b} = v'_j V_{jkl} b_k b_l, \tag{11}$$

$$\boldsymbol{b}'[(\boldsymbol{b}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{b}] = b'_j U_{jkl} b_k v_l, \tag{12}$$

where the tensor structure of the vertices in the wave-number-frequency representation is

$$V_{jkl} = \mathrm{i}(\delta_{jk}p_l + \delta_{jl}p_k) \qquad U_{jkl} = \mathrm{i}(\delta_{jl}p_k - \delta_{jk}p_l), \tag{13}$$

where the momentum $\mathbf{p}$ is flowing into the vertex via the auxiliary fields $\boldsymbol{v}'$ and $\boldsymbol{b}'$.

3. Renormalization

3.1. Divergences of the model

It can be shown [17] that for any fixed space dimension $d > 2$, the superficial UV divergences can exist only in the following one-particle irreducible (1PI) Green functions: $\Gamma^{vv'}$, $\Gamma^{bb'}$ and $\Gamma^{v'bb}$. They lead to local counter-terms of the form $\propto \boldsymbol{v}'\nabla^2\boldsymbol{v}$, $\propto \boldsymbol{b}'\nabla^2\boldsymbol{b}$ and $\propto \boldsymbol{v}'(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}$ which are already present in the action (5); therefore, the model is multiplicatively renormalizable (the analytic terms in k^2 proportional to g_{v20} and g_{v20} in (3) and (4) are not needed in this case, and the model can be formulated without them).

The situation is more complicated in the two-dimensional case, where additional UV divergences appear. They are related to the 1PI Green functions $\Gamma^{v'v'}$ and $\Gamma^{b'b'}$. In this situation the formulation of the model without local (analytic in k^2) terms cannot give, in general, the multiplicatively renormalizable model because the nonlocal terms of the action are not renormalized since the divergences produced by the loop integrals of the diagrams are always local in space and time (see, e.g., [13]). Thus, the simplest way to restore the renormalizability of the model (or to include the corresponding local counter-terms $\propto \boldsymbol{v}'\nabla^2\boldsymbol{v}'$ and $\propto \boldsymbol{b}'\nabla^2\boldsymbol{b}'$ in the renormalization) is to add the corresponding local terms to the force correlation functions. It is shown explicitly in (3) and (4). In language of classical hydrodynamics the forcing contribution $\propto k^2$ corresponds to the appearance of large eddies convected by small and active ones and it is represented by the local term of $\boldsymbol{v}'\nabla^2\boldsymbol{v}'$. In its analogy the term $\boldsymbol{b}'\nabla^2\boldsymbol{b}'$ is added to the magnetic forcing.

Thus, in two dimensions, the model (5) is renormalizable by the standard power-counting rules, and for limits $\epsilon \to 0, \delta \to 0$ possesses the ultraviolet (UV) divergences which are present in the five aforementioned 1PI Green functions. It means that the model is regularized using a combination of analytic and dimensional regularization with the parameters ϵ and $\delta = (d-2)/2$. As a result, the UV divergences appear as poles in ϵ, δ, and their following combinations: $2\epsilon + \delta$ and $(a+1)\epsilon + \delta$. The UV divergences may be removed by adding needed counter-terms to the basic action S_B which is obtained from the unrenormalized one (5) by the substitution of the renormalized parameters with the bare ones: $g_{v10} \to \mu^{2\epsilon} g_{v1}$, $g_{v20} \to \mu^{-2\delta} g_{v2}$, $g_{b10} \to \mu^{2a\epsilon} g_{b1}$, $g_{b20} \to \mu^{-2\delta} g_{b2}$, $\nu_0 \to \nu$, $u_0 \to u$, where μ is a scale-setting parameter having the same canonical dimension as the wave number.

In what follows, we shall work with, in our case the most convenient, minimal subtraction (MS) scheme, i.e., we are interested only in the singular (pole) parts of the divergent 1PI Green functions which are included in the renormalization constants. They give rise to the counter-terms added to the basic action to make the Green functions of the renormalized model UV finite. In our model, the counter-terms have the form

$$S_{\text{count}} = \int \mathrm{d}x \Big[\nu(1-Z_1)\boldsymbol{v}'\nabla^2\boldsymbol{v} + u\nu(1-Z_2)\boldsymbol{b}'\nabla^2\boldsymbol{b} + \frac{1}{2}(Z_4-1)u\nu^3 g_{v2}\mu^{-2\delta}\boldsymbol{v}'\nabla^2\boldsymbol{v}' + \frac{1}{2}(Z_5-1)u^2\nu^3 g_{b2}\mu^{-2\delta}\boldsymbol{b}'\nabla^2\boldsymbol{b}' + (1-Z_3)\boldsymbol{v}'(\boldsymbol{b}\cdot\nabla)\boldsymbol{b} \Big], \tag{14}$$

where the renormalization constants $Z_i, i = 1, 2, 4, 5$, renormalize the unrenormalized parameters $e_0 = \{g_{v10}, g_{v20}, g_{b10}, g_{b20}, \nu_0, u_0\}$, and the renormalization constant Z_3 renormalize the fields $\boldsymbol{b}$ and $\boldsymbol{b}'$. They are chosen to cancel the UV divergences appearing in the Green functions constructed using the basic action. The remaining fields $\boldsymbol{v}'$ and $\boldsymbol{v}$ are not renormalized due to the Galilean invariance of the model (5).

Renormalized Green functions are expressed in terms of the renormalized parameters

$$\begin{aligned} g_{v1} &= g_{v10}\mu^{-2\epsilon} Z_1^2 Z_2, & g_{v2} &= g_{v20}\mu^{2\delta} Z_1^2 Z_2 Z_4^{-1}, \\ g_{b1} &= g_{b10}\mu^{-2a\epsilon} Z_1 Z_2^2 Z_3^{-1}, & g_{b2} &= g_{b20}\mu^{2\delta} Z_1 Z_2^2 Z_3^{-1} Z_5^{-1}, \\ \nu &= \nu_0 Z_1^{-1}, & u &= u_0 Z_2^{-1} Z_1 \end{aligned} \tag{15}$$

appearing in the renormalized action S^R connected with the action (5) by the relation of multiplicative renormalization: $S^R\{e\} = S\{e_0\}$. The renormalized action S^R, which depends on the renormalized parameters $e(\mu)$, yields the renormalized Green functions without UV divergences. The RG is mainly concerned with the prediction of the asymptotic behaviour of correlation functions expressed in terms of anomalous dimensions γ_j by the use of β-functions, both defined via differential relations

$$\gamma_j = \mu \frac{\partial \ln Z_j}{\partial \mu}\bigg|_{e_0}, \qquad \beta_g = \mu \frac{\partial g}{\partial \mu}\bigg|_{e_0}, \qquad \text{with} \quad g \equiv \{g_{v1}, g_{v2}, g_{b1}, g_{b2}, u\}. \tag{16}$$

These definitions with expressions (15) yield the γ-functions

$$\begin{aligned} \gamma_{gv1} &= -2\gamma_1 - \gamma_2, & \gamma_{gb1} &= -\gamma_1 - 2\gamma_2 + \gamma_3, \\ \gamma_{gv2} &= -2\gamma_1 - \gamma_2 + \gamma_4, & \gamma_{gb2} &= -\gamma_1 - 2\gamma_2 + \gamma_3 + \gamma_5, \\ \gamma_\nu &= \gamma_1, \qquad \gamma_b = \tfrac{1}{2}\gamma_3, & \gamma_u &= -\gamma_1 + \gamma_2 \end{aligned} \tag{17}$$

and then the β-functions

$$\begin{aligned} \beta_{gv1} &= g_{v1}(-2\epsilon + 2\gamma_1 + \gamma_2), & \beta_{gb1} &= g_{b1}(-2a\epsilon + \gamma_1 + 2\gamma_2 - \gamma_3), \\ \beta_{gv2} &= g_{v2}(2\delta + 2\gamma_1 + \gamma_2 - \gamma_4), & \beta_{gb2} &= g_{b2}(2\delta + \gamma_1 + 2\gamma_2 - \gamma_3 - \gamma_5) \\ \beta_u &= u(\gamma_1 - \gamma_2). \end{aligned} \tag{18}$$

3.2. RG equations

Correlation functions of the fields are expressed in terms of scaling functions of the variable $s = (k/\mu)$, $s \in \langle 0, 1\rangle$. Then the asymptotic behaviour and the universality of MHD statistics stem from the existence of a stable IR fixed point. The continuous RG transformation is an operation linking a sequence of invariant parameters $\overline{g}(s)$ determined by the Gell–Mann–Low equations

$$\frac{\mathrm{d}\overline{g}(s)}{\mathrm{d}\ln s} = \beta_g\,(\overline{g}(s)) \text{ with the abbreviation } \overline{g} \equiv \{\overline{g}_{v1}, \overline{g}_{v2}, \overline{g}_{b1}, \overline{g}_{b2}, \overline{u}\}, \tag{19}$$

where the scaling variable s parametrizes the RG flow with the initial conditions $\overline{g}|_{s=1} \equiv g$ (the critical behaviour corresponds to IR limit $s \to 0$). The expression of the $\beta(\overline{g}(s))$ function is known in the framework of the δ, ϵ expansion (see (24) and also (18)). The fixed point $g^*(s \to 0)$ satisfies a system of equations $\beta_g(g^*) = 0$, while an IR stable fixed point, weakly dependent on initial conditions, is defined by positive definiteness of the real part of the matrix $\Omega = (\partial\beta_g/\partial g)|_{g^*}$ (the matrix of the first derivatives taken at the fixed point). In other words, a fixed point is stable if all the trajectories $g(s)$ in its vicinity approach the value of the fixed point.

The initial conditions $\overline{g}|_{s\to 1} = g$ of equations (19), dictated by a micromodel, are insufficient since our aim is the large-scale limit of statistical theory, where $g^* \equiv \overline{g}|_{s\to 0}$. As was mentioned already, the RG fixed point is defined by the equation

$$\beta(g^*) = 0. \tag{20}$$

For $\overline{g}(s)$ close to g^* we obtain a system of linearized equations

$$\left(I s \frac{\mathrm{d}}{\mathrm{d}s} - \Omega\right)(\overline{g} - g^*) = 0, \tag{21}$$

where I is a (5×5) unit matrix. Solutions of this system behave like $\overline{g} = g^* + \mathcal{O}(s^{\xi_j})$ if $s \to 0$. The exponents ξ_j are the elements of the diagonalized matrix $\Omega^{diag} = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$ and can be obtained as roots of the characteristic polynomial $\mathrm{Det}(\Omega - \xi I)$. The positive definiteness of Ω represented by the conditions $\mathrm{Re}_j(\xi) \geqslant 0$, $j = 1, 2, \ldots, 5$, is the test of the IR asymptotical stability of the discussed theory.

3.3. One-loop order calculation

In the standard MS scheme [35] the renormalization constants have the general form

$$Z_i = 1 - F_i P^{\delta,\epsilon}, \tag{22}$$

where the terms $P^{\delta,\epsilon}$ are given by the linear combinations of the poles and the amplitudes F_i are some functions of $g_{v1}, g_{v2}, g_{b1}, g_{b2}$ and u, but are independent of δ and ϵ. The amplitudes $F_i = F_i^{(1)} F_i^{(2)}$ are a product of two multipliers $F_i^{(1)}, F_i^{(2)}$. One of them, say $F_i^{(1)}$, is a multiplier originating from the divergent part of the Feynman diagrams, and the second one, $F_i^{(2)}$, is connected only with the tensor nature of the diagrams (see discussion in [29] for details).

It can be explained by the following simple example [29] (the example is taken from a problem with anisotropy, i.e., where another arbitrary unit vector $\mathbf{n}$ exists but the conclusions are the same). Consider an UV-divergent integral

$$I(\mathbf{k}, \mathbf{n}) \equiv n_i n_j k_l k_m \int \mathrm{d}^d\mathbf{q} \frac{1}{(q^2+m^2)^{1+2\delta}} \left(\frac{q_i q_j q_l q_m}{q^4} - \frac{\delta_{ij} q_l q_m + \delta_{il} q_j q_m + \delta_{jl} q_i q_m}{3q^2} \right)$$

(summations over repeated indices are implied) where m is an infrared mass. It can be simplified in the following way:

$$I(\mathbf{k}, \mathbf{n}) \equiv n_i n_j k_l k_m S_{ijlm} \int_0^\infty \mathrm{d}q^2 \frac{q^{2\delta}}{2(q^2+m^2)^{1+2\delta}},$$

where

$$S_{ijlm} = \frac{S_d}{d(d+2)} (\delta_{ij}\delta_{lm} + \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl} - \frac{(d+2)}{3} (\delta_{ij}\delta_{lm} + \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})),$$

$$\int_0^\infty \mathrm{d}q^2 \frac{q^{2\delta}}{2(q^2+m^2)^{1+2\delta}} = \frac{\Gamma(\delta+1)\Gamma(\delta)}{2m^{2\delta}\Gamma(2\delta+1)},$$

and $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface of unit of the d-dimensional sphere. The purely UV divergent part manifests itself as the pole in $2\delta = d - 2$; therefore, we find

$$\text{UV div. part of } I = \frac{1}{2\delta}\left(F_1^{(2)} k^2 + F_2^{(2)} (\mathbf{nk})^2\right),$$

where $F_1^{(2)} = F_2^{(2)}/2 = (1-d)S_d/3d(d+2)$ $\left(F_1^{(1)} = F_2^{(1)} = 1\right)$. It has to be mentioned that in spite of the above simple example in our calculation we shall introduce the needed IR regularization by restriction on the interval of integrations.

In the standard MS scheme one puts $d = 2$ in $F_1^{(2)}, F_2^{(2)}$; therefore, the d-dependence of these multipliers is ignored. As was discussed in [29], for the theories with vector fields and, consequently, with tensor diagrams, where the sign of the values of fixed points and/or their stability depend on the dimension d, the procedure, which eliminates the dependence of multipliers of the type $F_1^{(2)}, F_2^{(2)}$ on d, is not completely correct because one is not able to control the stability of the fixed point when $d = 3$. Therefore, in [29] it was proposed to slightly modify the MS scheme in such a way to keep the d-dependence of F in renormalization constants Z_i. Then the subsequent calculations of the RG functions (β-functions and anomalous dimensions γ_i) allow one to arrive at the results which are in qualitative agreement with the results obtained in the framework of the simple analytical regularization scheme, i.e., one is able to obtain the fixed point which is not stable for $d = 2$, but whose stability is restored for a borderline dimension $2 < d_c < 3$. In what follows, it will be shown that it is really our case; thus, we shall apply this modified MS scheme in our calculations.

Now we can return and continue with RG analysis. Using the RG routine the anomalous dimensions $\gamma_j(g_{v1}, g_{v2}, g_{b1}, g_{b2})$ can be extracted from one-loop diagrams. Thus, the extraction of the UV-divergent parts from one-loop diagrams gives Z-constants in the form

$$
\begin{aligned}
Z_1 &= 1 + \frac{S_d}{(2\pi)^d}\left[u\lambda_5\left(\frac{g_{v2}}{2\delta} - \frac{g_{v1}}{2\epsilon}\right) + \lambda_6\left(\frac{g_{b2}}{2\delta} - \frac{g_{b1}}{2a\epsilon}\right)\right],\\
Z_2 &= 1 + \frac{S_d}{(2\pi)^d(u+1)}\left[\lambda_1\left(\frac{g_{v2}}{2\delta} - \frac{g_{v1}}{2\epsilon}\right) + \lambda_3\left(\frac{g_{b2}}{2\delta} - \frac{g_{b1}}{2a\epsilon}\right)\right],\\
Z_3 &= 1 + \frac{S_d}{(2\pi)^d}\lambda_7\left(\frac{g_{v1}}{2\epsilon} - \frac{g_{v2}}{2\delta} - \frac{g_{b1}}{2a\epsilon} + \frac{g_{b2}}{2\delta}\right),\\
Z_4 &= 1 + \frac{S_d}{(2\pi)^d}\frac{\lambda_4}{g_{v2}}\left(\frac{ug_{v1}^2}{2\delta + 4\epsilon} + \frac{2ug_{v1}g_{v2}}{2\epsilon} - \frac{ug_{v2}^2}{2\delta} + \frac{g_{b1}^2}{2\delta + 4a\epsilon} + \frac{2g_{b1}g_{b2}}{2a\epsilon} - \frac{g_{b2}^2}{2\delta}\right),\\
Z_5 &= 1 + \frac{S_d}{(2\pi)^d}\frac{\lambda_2}{(u+1)g_{b2}}\left(\frac{g_{v1}g_{b1}}{2\delta + 2\epsilon(1+a)} + \frac{g_{v1}g_{b2}}{2\epsilon} + \frac{g_{v2}g_{b1}}{2a\epsilon} - \frac{g_{v2}g_{b2}}{2\delta}\right),
\end{aligned}
\tag{23}
$$

and, in consequence, the lowest order γ-functions are

$$
\begin{aligned}
\gamma_1 &= \widetilde{S}_d\,(u\lambda_5 g_v + \lambda_6 g_b)\,, & \gamma_2 &= \widetilde{S}_d\frac{(\lambda_1 g_v + \lambda_3 g_b)}{u+1},\\
\gamma_3 &= \widetilde{S}_d\lambda_7(-g_v + g_b), & \gamma_4 &= \widetilde{S}_d\frac{\lambda_4}{g_{v2}}\left(ug_v^2 + g_b^2\right),\\
\gamma_5 &= \widetilde{S}_d\frac{\lambda_2}{(1+u)}\frac{g_v g_b}{g_{b2}},
\end{aligned}
\tag{24}
$$

where $\widetilde{S}_d = S_d/(2\pi)^d$, S_d denotes the d-dimensional sphere $S_d = 2\pi^{d/2}/\Gamma(d/2)$, $g_v \equiv g_{v1} + g_{v2}$, $g_b \equiv g_{b1} + g_{b2}$, and λ-coefficients depend only on the dimension d:

$$
\begin{aligned}
\lambda_1 &= \frac{d-1}{2d}, \quad \lambda_2 = \frac{d-2}{2d}, \quad \lambda_3 = \frac{d-3}{2d}, \quad \lambda_4 = \frac{d^2-2}{4d(d+2)},\\
\lambda_5 &= \frac{d-1}{4(d+2)}, \quad \lambda_6 = \frac{d^2+d-4}{4d(d+2)}, \quad \lambda_7 = \frac{1}{d(d+2)}.
\end{aligned}
\tag{25}
$$

Substituting (24) into β-functions (18) one can obtain β-functions in the one-loop order approximation. Note that in two dimensions the γ-functions are

$$
\begin{aligned}
\gamma_1^{(2)} &= \frac{1}{32\pi}\,(ug_v + g_b)\,, \quad \gamma_2^{(2)} = \frac{1}{8\pi}\frac{(g_v - g_b)}{(u+1)}, \quad \gamma_3^{(2)} = \frac{1}{16\pi}(-g_v + g_b),\\
\gamma_4^{(2)} &= \frac{1}{32\pi}\frac{\left(ug_v^2 + g_b^2\right)}{g_{v2}}, \quad \gamma_5^{(2)} = 0
\end{aligned}
\tag{26}
$$

and, in correspondence with [27] $Z_5 = 1$, which is a specific property of the two-dimensional MHD turbulence because there are no UV divergences in 1PI Green's function $\Gamma^{b'b'}$ in the one-loop approximation. Here we emphasize that in the general case of d dimensions $\gamma_5 \neq 0$ and $Z_5 \neq 1$.

4. Fixed points

4.1. Case of passive vector admixture

Here we briefly consider the case when the magnetic field can be treated as a passive vector field in the developed HD turbulence. Notation of the 'passive' magnetic field means that the Lorentz force acting on conductive fluid can be neglected at large spatial scales; thus, the Lorentzian term $(\mathbf{b}\cdot\nabla)\mathbf{b}$ in the Navier–Stokes equation can be omitted. Just then the vertex function $\Gamma^{v'bb}$ is finite and the term containing Z_3 in S_{count} does not exist. Therefore, the magnetic field is not renormalized and $\gamma_3 = 0$. Furthermore, some diagrams of $\Gamma^{v'v}$, $\Gamma^{v'v'}$ and $\Gamma^{b'b}$ containing the vertex $\Gamma^{v'bb}$ can be omitted and Z-constants as well as γ-functions are reduced. Resulting γ-functions take the form

$$\gamma_1 = \widetilde{S}_d u\lambda_5 g_v, \qquad \gamma_2 = \widetilde{S}_d\lambda_1\frac{g_v}{u+1},$$
$$\gamma_4 = \widetilde{S}_d\lambda_4\frac{u}{g_{v2}}g_v^2, \qquad \gamma_5 = \widetilde{S}_d\frac{\lambda_2}{(1+u)}\frac{g_v g_b}{g_{b2}}. \tag{27}$$

Substituting the γ-functions (27) and $\gamma_3 = 0$ into β-equations (18) one obtains a system of four nonlinear equations $\beta_{gv1} = \beta_{gv2} = \beta_{gb1} = \beta_{gb2} = 0$ for g_i and one equation $\beta_u = 0$ for u. The last one gives $u^* = 0$, or, the nonzero universal inverse Prandtl number,

$$u^* = \frac{1}{2}\left(\sqrt{\frac{16+9d}{d}} - 1\right). \tag{28}$$

In the first case of $u^* = 0$, one obtains only two fixed points (with zeroth g_{b1}^*, g_{b2}^*):

(1) $g_{v1}^* = 0$, $g_{v2}^* = -2\delta/\lambda_1\widetilde{S}_d$, which is non-physical (negative), and
(2) $g_{v1}^* = 2\epsilon/\lambda_1\widetilde{S}_d$, $g_{v2}^* = 0$, which in unstable.

Let u is given by (28). Then apart from the Gaussian fixed point $g_{v1}^* = g_{v2}^* = g_{b1}^* = g_{b2}^* = 0$, with no fluctuation effect on the large-scale asymptotics, there are following fixed points with $g_{b2}^* = 0$:

$$(1^*)\quad g_{v1}^* = 0, \qquad g_{v2}^* = -\frac{2(d-2)d^2(u^*+1)}{2d^2-3d+2}\widetilde{S}_d^{-1}, \qquad g_{b1}^* = 0;$$
$$(2^*)\quad g_{v1}^* = \frac{4\epsilon d(u^*+1)}{3(d-1)}\widetilde{S}_d^{-1}, \qquad g_{v2}^* = 0, \qquad g_{b1}^* = 0;$$
$$(3^*)\quad g_{v1}^* = \frac{4\epsilon(3d^3+d^2(4\epsilon-9)-6d(\epsilon-1)+4\epsilon)(u^*+1)}{9(d+2\epsilon-2)(d-1)^2}\widetilde{S}_d^{-1},$$
$$g_{v2}^* = \frac{8\epsilon^2(d^2-2)(u^*+1)}{9(d+2\epsilon-2)(d-1)^2}\widetilde{S}_d^{-1}, \qquad g_{b1}^* = 0.$$

Next three fixed points are the same as the last (1^*)–(3^*) with different g_{b2}^*:

$(1a^*)$ $g_{b2}^* = (d^2-2)/d(d-2)$;
$(2a^*) \equiv (3a^*) g_{b2}^* = 3(d-1)(d+2\epsilon-2/2(d-2)\epsilon$.

The points $(2a^*)$ and $(3a^*)$ have the same g_{b2}^* because g_{v1}^* of the point (2^*) is equal to the sum $(g_{v1}^* + g_{v2}^*)$ of the point (3^*). Note that g_{b2}^* has discontinuity at $d = 2$.

The 'thermal' point (1^*) is generated by short-range correlations of the random force [27] and has negative g_{v2}^*. The second fixed point (2^*) is unstable. The physical meaning has the third 'kinetic' point (3^*) whose parameter $\{g_1, g_2, u\}$ dependence on the dimension d is shown in figure 1. for physical value of $\epsilon = 2$.

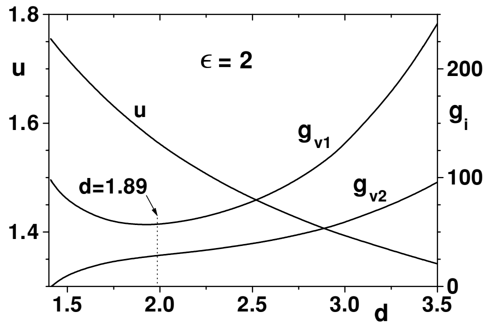

Figure 1. Dependence of the parameters $\{g_{v1}, g_{v2}, u\}$ on the dimension d for $\epsilon = 2$ at the kinetic fixed point (29).

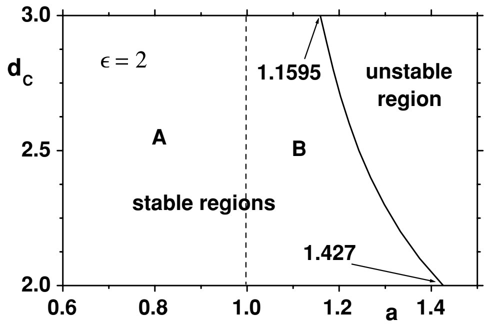

Figure 2. Stability regions of the kinetic point and the critical dimension d_c dependence on the parameter a. The region A spreads down to $a = 0$.

Setting $\epsilon = 2$ and u^* from (28) one obtains

$$g_{v1}^* = \frac{(2\pi)^d}{S_d}\frac{8(u^*+1)(3d^3 - d^2 - 6d + 8)}{9(d-1)^2(d+2)} \qquad g_{v2}^* = \frac{(2\pi)^d}{S_d}\frac{32(u^*+1)(d^2-2)}{9(d-1)^2(d+2)}. \qquad (29)$$

In this case $g_{v1}^* + g_{v2}^* \equiv g_v^* = (2\pi)^d 8d(u^*+1)/3(d-1)S_d$. Detailed numerical calculations have shown that the region of stability of this point is limited by the value of parameter $a < 1$ and this limiting value does not depend on the dimension d. This stable region is denoted as region A in figure 2.

4.2. Case of active vector admixture

In the full self-consistent system, the RG equations yield besides the known fixed point in the kinetic regime also the nontrivial magnetic fixed point. If both are stable in the same region of parameters, then the choice between two possible critical regimes will depend on the initial conditions for the RG equations, i.e. critical behaviour of the system is non-universal.

4.2.1. Kinetic fixed point. The nontrivial stable kinetic fixed point of the RG equations has been found to be the same as in the previous case of passive magnetic field admixture because the β-functions β_{gv1}, β_{gv2} are the same for zero g_{b1}, g_{b2}. Only difference was found in the

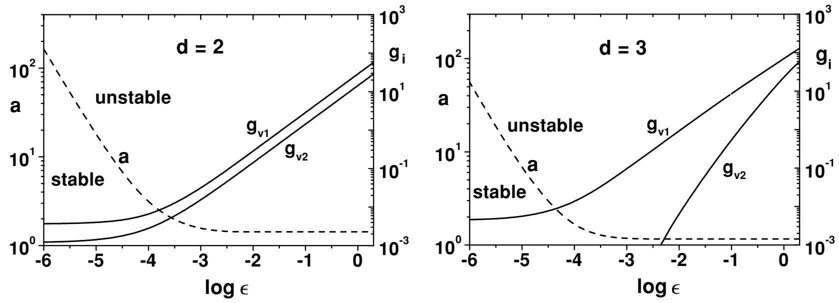

Figure 3. Dependence of the parameters $\{g_{v1}, g_{v2}, u\}$ on the value of ϵ for two and three dimensions at the kinetic fixed point in the general case. The dashed line shows the critical value of a at the stability region limit.

stability region in dependence on the parameter a: the stable region is enlarged by a new region B unlike the case of passive magnetic field admixture, see figure 2. The critical dimension d_c continuously decreases from 3 to 2 in dependence on the value of the parameter a from the interval $\langle 1.1595, 1.427\rangle$. It confirms the results of [27] that the stability of the kinetic scaling regime is strongly affected by the behaviour of magnetic fluctuations.

Figure 3 shows values of the charges g_{v1}, g_{v2} which continuously depend on the value of nonzero $\epsilon \leqslant 2$, for two special behaviour cases of d equal to 2 and 3. The right axle corresponds to the physical value of $\epsilon = 2$. While both the charges remain nonzero (positive) for $d = 2$, in three dimensions one of them, g_{v2}, rapidly decreases for $\epsilon \to 0$. The stable and unstable regions depend on the parameter a and the critical value of a remains the same for $\epsilon = 2$ following from figure 2, or greater for $\epsilon < 2$ (the critical a increases for $\epsilon \to 0$).

4.2.2. Magnetic fixed point. We have shown in (26) that in two dimensions the function γ_5 vanishes and then both functions β_{gb1} and β_{gb2} contain the same linear combination of γ functions. Thus, at least one of the magnetic charges (g_{b1}, g_{b2}) must be zero at the fixed point. But in the other dimensions this restriction does not take place.

Here we restrict ourselves only by finding the nontrivial magnetic fixed point. In [27] it was mentioned that it is characterized by zero g_{v1}^* and u^*. Therefore, the set of five equations of zero β-functions (18) is reduced to three equations. Applying $g_{v1} = u = 0$ in (18), (24) and (25) one obtains the set

$$\begin{aligned} &a_1 g_{v2} + a_2 g_{v2}^2 + a_3 g_{v2} g_b - a_4 g_b^2 = 0,\\ &-A_0 + a_5 g_{v2} + a_6 g_b = 0,\\ &a_1 g_{b2} + a_5 g_{v2} g_{b2} + a_6 g_{b2} g_b - a_7 g_{v2} g_b = 0, \end{aligned} \tag{30}$$

where

$$\begin{aligned} &A_0 = \frac{2a\epsilon}{S_d}, \qquad a_1 = \frac{2(d-2)}{S_d}, \qquad a_2 = \frac{(d-1)}{2d},\\ &a_3 = \frac{(d^2-5)}{d(d+2)}, \qquad a_4 = \frac{(d^2-2)}{4d(d+2)}, \qquad a_5 = \frac{(d^2+d-1)}{d(d+2)},\\ &a_6 = \frac{(5d^2-3d-32)}{4d(d+2)}, \qquad a_7 = \frac{(d-2)}{2d}. \end{aligned} \tag{31}$$

Positive coefficients a_1, a_7 vanishes at $d = 2$, a_3 and a_6 are positive for $d > 2.236$ and $d > 2.848$, respectively. The set (30) can be analytically solved with respect to g_{v2}, g_{b1}, g_{b2}.

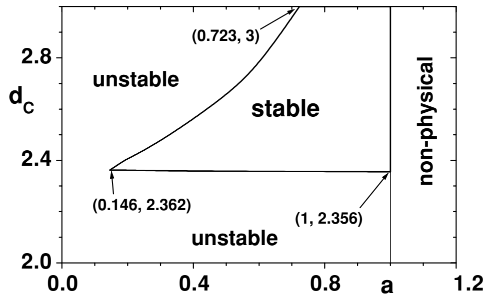

Figure 4. The stability region of the magnetic fixed point in the plane of $\{d, a\}$ for the physical value of $\epsilon = 2$.

Because all g_i must be positive, the system (30) with $g_{v1} = u = 0$ gives the only solution

$$g_{v2} = \frac{A_0 - a_6 g_b}{a_5}, \qquad g_{b1} = g_b - \frac{g_b(a_6 g_b - A_0) - a_5 a_7}{a_5(a_1 + 2a_6 g_b - A_0)},$$
$$g_{b2} = \frac{g_b(a_6 g_b - A_0) - a_5 a_7}{a_5(a_1 + 2a_6 g_b - A_0)}, \tag{32}$$

where

$$g_b = \frac{-a_1 a_5 a_6 + a_3 a_5 A_0 - 2a_2 a_6 A_0 + a_5\sqrt{D}}{2\left(a_4 a_5^2 + a_3 a_5 a_6 - a_2 a_6^2\right)},$$
$$D = a_1^2 a_6^2 + 4a_1 a_4 a_5 A_0 + 2a_1 a_3 a_6 A_0 + a_3^2 A_0^2 + 4a_2 a_4 A_0^2. \tag{33}$$

Note that the parameters a and ϵ appear in the solution only as the product $a\epsilon$ in A_0. The physical value is restricted by the inequality $a\epsilon \leqslant 2$. The denominator in expression (33) for g_b is zero at $d_0 = 2.2628$ (and it is positive for $d > d_0$); therefore, at d_0 we can expect discontinuity and/or divergence. Numerical analysis of expressions (32) shows that all g_i have a discontinuity at d_0, and a physical solution cannot exist for any a, ϵ if $d \leqslant d_0$. The stability region of the magnetic fixed point and the corresponding critical dimension d_c was determined numerically and it is shown in figure 4.

Figure 5 demonstrates the dependence of the charges g_{v2}, g_{b1}, g_{b2} on the dimension d for several values of the parameter a. First, we have found that g_{v2}, g_{b2} tend to infinity at the limit value d_0. For increasing dimension d from 2 up to d_0 the charge g_{b2} increases from a small positive value up to infinity at d_0 and, therefore, g_{v2} decreases here from a small negative value to minus infinity at d_0 (because $g_{v2} \propto -a_6 g_b$ and both a_6 and g_b are negative in these dimensions). The charge g_{b1} rapidly decreases to zero at $d = 2.352$ for decreasing d and continues to minus infinity at d_0. These limiting values are in correspondence with the numerical calculation of the stability region—the system loses stability for the critical dimension d_c lower than approximately 2.36 for arbitrary parameter a.

5. Discussion and conclusions

In this paper we revised the calculations of stability ranges of developed magnetohydrodynamic turbulence in the frame of the double expansion scheme. The modified standard minimal subtraction scheme [29] has been used in the dimension region of $d \geqslant 2$ up to $d = 3$ in both cases of the magnetic field treated as a passive as well as active vector admixture. We confirm the existence of the known 'kinetic' fixed point (corresponding to the Kolmogorov

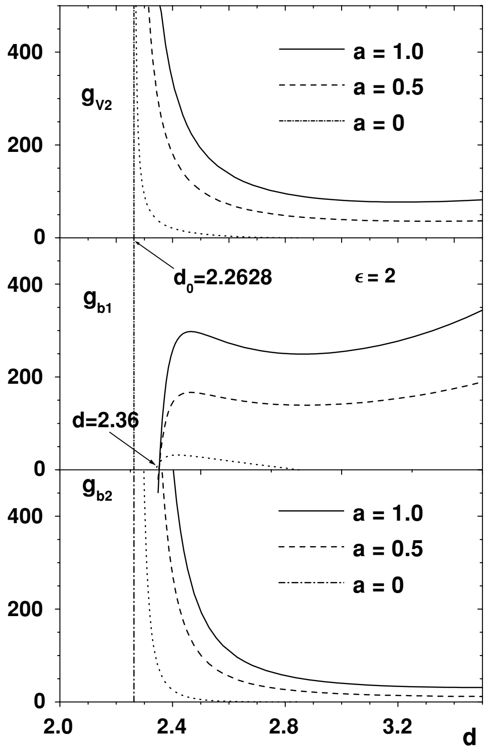

Figure 5. Dependence of the parameters $\{g_{v2}, g_{b1}, g_{b2}\}$ on the dimension d for $\epsilon = 2$ at the magnetic fixed point (32) for $a = 1, 0.5, 0$. All g_i have discontinuity at $d = 2.2628$ (chained vertical line).

scaling regime) which is the same in both the considered cases and the only difference is in the stability region: the critical dimension d_c is achieved for a slightly higher value of the a-parameter of a magnetic forcing in the case of active magnetic field. The limit value of the inverse Prandtl number at $d = 3$ restores the value of $u = 1.393$ which is known from the usual ϵ-expansion, and it fluently rises to $u = 1.562$ at $d = 2$ (figure 1).

It was believed earlier that in the double expansion being defined for the space dimension to be closed to 2 the results obtained in two dimensions cannot be applicable to the opposite dimension interval end closed to 3. Here we have showed that the double expansion in exact d-dimensional formulation gives some critical dimension d_c above which the scaling regime is governed by the competition of the stable kinetic and magnetic fixed points which exist in three dimensions.

A new nontrivial result of the present paper is connected with derivation of the exact analytical expression for the nontrivial 'magnetic' stable fixed point with $u = g_{v1} = 0$ but nonzero g_{v2}, g_{b1} and g_{b2} as well as specification of the borderline dimension d_c. A physical region of the RG fixed point lies below the $a\epsilon = 2$ line, see figure 4. This point completely loses stability below the critical value of dimension $d_c = 2.36$ (independently of the a-parameter) and also below the value of $a_c = 0.146$ (independently of the dimension). Thus we confirm, in particular, that thermal fluctuations of the magnetic scaling regime may occur, and in comparison with earlier results our value of the borderline dimension ($d_c = 2.36$) is significantly lower than that in the ϵ-expansion [10] ($d_c = 2.85$) and rather lower than that in the ‘modified’ double expansion introduced in [27] ($d_c = 2.46$), but it is rather higher than the value ($d_c = 2.2$) calculated in the frame of McComb’s renormalization [21].

Note that the stability of any regime determines the concrete Alfven ratio r_A (ratio of the kinetic and magnetic energy density in the MHD turbulence, see [22, 23], for example). Once the stationary scaling regime becomes and stands, the Alfven ratio is fixed (i.e., it means that the fixed point is reached in the field RG terminology). Thus, the injected energy necessary to steady the stationary scaling regime must have a specific value, or, in other words, all ‘coupling constants’ g_i are fixed in the scaling regime with values which are dependent on the dimension d. In a like manner the inverse Prandtl number $u \equiv \eta/\nu$ (η is the magnetic resistivity) is thus fixed. Verma [23] has obtained $\eta/\nu = 0.85/0.36 = 2.36$ in three dimensions for large $r_A \approx 5000$ (corresponding to the region of the kinetic regime) and for zeroth normalized cross-helicity. For smaller r_A this ratio decreases to 0.69 for $r_A = 1$, and both η and ν vary approximately as $d^{-1/2}$ [22]. We have mentioned above that in our double expansion calculation at the kinetic point we have fixed the ratio $u \equiv \eta/\nu$ with its d-dependence shown in figure 1. The magnetic fixed point is characterized by decreasing the value of u to zero which is in correspondence with the results of [23]: his calculation gives for decreasing r_A (magnetic regime) in three dimensions also the decreasing value of η/ν as one can expect at the magnetic fixed point.

Acknowledgments

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